

IDENTITIES OF SYMMETRY FOR HIGHER-ORDER q -BERNOULLI POLYNOMIALS

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ABSTRACT. Recently, the higher-order Carlitz's q -Bernoulli polynomials are represented as q -Volkenborn integral on \mathbb{Z}_p by Kim. A question was asked in [?] as to finding the extended formulae of symmetries for Bernoulli polynomials which are related to Carlitz q -Bernoulli polynomials. In this paper, we give some new identities of symmetry for the higher-order Carlitz's q -Bernoulli polynomials which are derived from multivariate q -Volkenborn integrals on \mathbb{Z}_p . We note that they are a partial answer to that question.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers and the completion of algebraic closure of \mathbb{Q}_p . The p -adic absolute value in \mathbb{C}_p is normalized so that $|p|_p = p^{-1}$. Let q be an indeterminate in \mathbb{C}_p with $|1 - q|_p < p^{-\frac{1}{p-1}}$. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient,

$$F_f : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{Z}_p \quad \text{by} \quad F_f(x, y) = \frac{f(x) - f(y)}{x - y},$$

have a limit $l = f'(a)$ as $(x, y) \rightarrow (a, a)$. If f is uniformly differentiable on \mathbb{Z}_p , we denote this property by $f \in UD(\mathbb{Z}_p)$.

For $f \in UD(\mathbb{Z}_p)$, the q -Volkenborn integral is defined by Kim to be

$$(1) \quad I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x,$$

where $[x]_q = \frac{1-q^x}{1-q}$, (see [?, ?, ?]).

From (1), we note that

$$(2) \quad qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q} f'(0)$$

where $f_1(x) = f(x+1)$.

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As is well known, the Bernoulli polynomials are defined by the generating function to be

$$(3) \quad \frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

By (3), we get

$$(4) \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases} \text{ and } B_0 = 1.$$

In [?], Carlitz defined q -Bernoulli numbers as follows :

$$(5) \quad \beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing β_q^i by $\beta_{i,q}$.

From (4) and (5), we note that $\lim_{q \rightarrow 1} \beta_{n,q} = B_n$.

The q -Bernoulli polynomials are given by

$$(6) \quad \begin{aligned} \beta_{n,q}(x) &= \sum_{l=0}^n \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l} \\ &= \left(q^x \beta_q + [x]_q \right)^n, \quad (n \geq 0), \quad (\text{see [?, ?, ?, ?]}). \end{aligned}$$

In [?], Kim proved that Carlitz q -Bernoulli polynomials can be written by q -Volkenborn integral on \mathbb{Z}_p as follows :

$$(7) \quad \begin{aligned} \beta_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y) \\ &= \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l d\mu_q(x). \end{aligned}$$

Thus, by (7), we get

$$(8) \quad \beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x), \quad (n \geq 0).$$

From (2), we note that

$$(9) \quad q \int_{\mathbb{Z}_p} [x+1]_q^n d\mu_q(x) - \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \begin{cases} q-1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

By (7), (8) and (9), we see that

$$(10) \quad \beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Let

$$(11) \quad I_1(f) = \lim_{q \rightarrow 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) \\ = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see } [?, ?, ?]).$$

Then, by (2), we get

$$(12) \quad I_1(f_1) - I_1(f) = f'(0).$$

Let us take $f(x) = e^{tx}$. Then we have

$$(13) \quad \int_{\mathbb{Z}_p} e^{xt} d\mu_1(x) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

and

$$(14) \quad \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \left(\frac{t}{e^t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

For $r \in \mathbb{N}$, the higher-order Bernoulli polynomials are defined by the generating function to be

$$(15) \quad \left(\frac{t}{e^t - 1} \right)^r e^{xt} = \underbrace{\left(\frac{t}{e^t - 1} \right) \times \cdots \times \left(\frac{t}{e^t - 1} \right)}_{r\text{-times}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}.$$

By (14), we get

$$(16) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x+y_1+\cdots+y_r)t} d\mu_1(y_1) \cdots d\mu_1(y_r) = \left(\frac{t}{e^t - 1} \right)^r e^{xt} \\ = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}.$$

In [?, ?], Carlitz introduced the q -extension of higher-order Bernoulli polynomials as follows :

$$(17) \quad \beta_{n,q}^{(r)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \left(\frac{l+1}{[l+1]_q} \right)^r,$$

where $n \geq 0$ and $r \in \mathbb{N}$.

Note that $\lim_{q \rightarrow 1} \beta_{n,q}^{(r)}(x) = B_n^{(r)}(x)$.

From (16), we note that

$$(18) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + y_1 + \cdots + y_r)^n d\mu_1(y_1) \cdots d\mu_1(y_r) = B_n^{(r)}(x),$$

where $n \geq 0$ and $r \in \mathbb{N}$.

In this paper, we consider q -extensions of (17) which are related to higher-order Carlitz's q -Bernoulli polynomials. The purpose of this paper is to give some new and interesting identities of symmetry for the higher-order Carlitz's q -Bernoulli polynomials which are derived from multivariate q -Volkenborn integral on \mathbb{Z}_p .

2. IDENTITIES OF SYMMETRY FOR HIGHER-ORDER q -BERNOULLI POLYNOMIALS

In the sense of q -extension of (18), we observe the following equation (19)

$$(19) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + y_1 + \cdots + y_r]_q^n d\mu_q(y_1) \cdots d\mu_q(y_r) \\ = \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \left(\frac{l+1}{[l+1]_q} \right)^r.$$

Thus, by (17) and (19), we get

$$(20) \quad \beta_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + y_1 + \cdots + y_r]_q^n d\mu_q(y_1) \cdots d\mu_q(y_r),$$

where $n \geq 0$ and $r \in \mathbb{N}$.

Let us consider the generating function of $\beta_{n,q}^{(r)}(x)$ as follows :

$$(21) \quad \sum_{n=0}^{\infty} \beta_{n,q}^{(r)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x+y_1+\cdots+y_r]_q t} d\mu_q(y_1) \cdots d\mu_q(y_r).$$

For $w_1, w_2 \in \mathbb{N}$, we have

$$(22) \quad \frac{1}{[w_1]_q^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l]_q t} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\ = \lim_{N \rightarrow \infty} \left(\frac{1}{[w_1]_q [p^N]_{q^{w_1}}} \right)^r \sum_{y_1, \dots, y_r=0}^{p^N-1} e^{[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l]_q t} q^{w_1(y_1 + \cdots + y_r)} \\ = \lim_{N \rightarrow \infty} \left(\frac{1}{[w_1]_q [w_2 p^N]_{q^{w_1}}} \right)^r \\ \times \sum_{y_1, \dots, y_r=0}^{w_2 p^N-1} e^{[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l]_q t} q^{w_1(y_1 + \cdots + y_r)} \\ = \lim_{N \rightarrow \infty} \left(\frac{1}{[w_1 w_2 p^N]_q} \right)^r \\ \times \sum_{i_1, \dots, i_r=0}^{w_2-1} \sum_{y_1, \dots, y_r=0}^{p^N-1} e^{[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 i_l + w_1 w_2 y_l)]_q t} q^{w_1 \sum_{l=1}^r (i_l + w_2 y_l)}.$$

Thus, by (22), we get

$$\begin{aligned}
 (23) \quad & \frac{1}{[w_1]_q^r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \\
 & \times \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 y_l)]_q} d\mu_{q^{w_1}}(y_1) \dots d\mu_{q^{w_1}}(y_r) \\
 & = \lim_{N \rightarrow \infty} \left(\frac{1}{[w_1 w_2 p^N]_q} \right)^r \sum_{j_1, \dots, j_r=0}^{w_1-1} \sum_{i_1, \dots, i_r=0}^{w_2-1} \\
 & \times \sum_{y_1, \dots, y_r=0}^{p^N-1} e^{[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 i_l + w_1 w_2 y_l)]_q} q^{\sum_{l=1}^r (w_1 i_l + w_2 j_l + w_1 w_2 y_l)}.
 \end{aligned}$$

By the same method as (23), we get

$$\begin{aligned}
 (24) \quad & \frac{1}{[w_2]_q^r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r j_l} \\
 & \times \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{[w_1 w_2 x + \sum_{l=1}^r (w_1 j_l + w_2 y_l)]_q} d\mu_{q^{w_2}}(y_1) \dots d\mu_{q^{w_2}}(y_l) \\
 & = \lim_{N \rightarrow \infty} \left(\frac{1}{[w_1 w_2 p^N]_q} \right)^r \sum_{j_1, \dots, j_r=0}^{w_2-1} \sum_{i_1, \dots, i_r=0}^{w_1-1} \\
 & \times \sum_{y_1, \dots, y_r=0}^{p^N-1} e^{[w_1 w_2 x + \sum_{l=1}^r (w_1 j_l + w_2 i_l + w_1 w_2 y_l)]_q} q^{\sum_{l=1}^r (w_2 i_l + w_1 j_l + w_1 w_2 y_l)}.
 \end{aligned}$$

Therefore, by (23), we obtain the following theorem.

Theorem 1. For $w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}
 & \frac{1}{[w_1]_q^r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \\
 & \times \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 y_l)]_q} d\mu_{q^{w_1}}(y_1) \dots d\mu_{q^{w_1}}(y_r) \\
 & = \frac{1}{[w_2]_q^r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r j_l} \\
 & \times \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{[w_1 w_2 x + \sum_{l=1}^r (w_1 j_l + w_2 y_l)]_q} d\mu_{q^{w_2}}(y_1) \dots d\mu_{q^{w_2}}(y_r).
 \end{aligned}$$

It is easy to show that

$$\begin{aligned}
 (25) \quad & [w_1 w_2 x + w_2 (j_1 + \dots + j_r) + w_1 (y_1 + \dots + y_r)]_q \\
 & = [w_1]_q \left[w_2 x + \frac{w_2}{w_1} (j_1 + \dots + j_r) + (y_1 + \dots + y_r) \right]_{q^{w_1}}.
 \end{aligned}$$

Therefore, by (20), Theorem 1 and (25), we obtain the following corollary, and theorem.

Corollary 2. For $n \geq 0$ and $w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}
& [w_1]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \\
& \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \frac{w_2}{w_1} (j_1 + \cdots + j_r) + y_1 + \cdots + y_r \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
& = [w_2]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r j_l} \\
& \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_1 x + \frac{w_1}{w_2} (j_1 + \cdots + j_r) + y_1 + \cdots + y_r \right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y_1) \cdots d\mu_{q^{w_2}}(y_r).
\end{aligned}$$

Theorem 3. For $n \geq 0$ and $w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}
& [w_1]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2(j_1 + \cdots + j_r)} \beta_{n, q^{w_1}}^{(r)} \left(w_2 x + \frac{w_2}{w_1} (j_1 + \cdots + j_r) \right) \\
& = [w_2]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1(j_1 + \cdots + j_r)} \beta_{n, q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2} (j_1 + \cdots + j_r) \right).
\end{aligned}$$

Remark. Let $w_2 = 1$. Then we have

$$\beta_{n, q}^{(r)}(w_1 x) = [w_1]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{j_1 + \cdots + j_r} \beta_{n, q^{w_1}}^{(r)} \left(x + \frac{j_1 + \cdots + j_r}{w_1} \right).$$

By (20), we see that

(26)

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \frac{w_2}{w_1} (j_1 + \cdots + j_r) + y_1 + \cdots + y_r \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
& = \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i) \sum_{l=1}^r j_l} \\
& \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \sum_{l=1}^r y_l \right]_{q^{w_1}}^{n-i} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
& = \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i) \sum_{l=1}^r j_l} \beta_{n-i, q^{w_1}}^{(r)}(w_2 x).
\end{aligned}$$

From Corollary 2 and (26), we have

$$\begin{aligned}
 (27) \quad & [w_1]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \\
 & \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
 & = \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} \\
 & \quad \times [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i) \sum_{l=1}^r j_l} \beta_{n-i, q^{w_1}}^{(r)}(w_2 x) \\
 & = \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} \beta_{n-i, q^{w_1}}^{(r)}(w_2 x) \\
 & \quad \times \sum_{j_1, \dots, j_r=0}^{w_1-1} [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i+1) \sum_{l=1}^r j_l} \\
 & = \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i, q^{w_1}}^{(r)}(w_2 x) \\
 & \quad \times \sum_{j_1, \dots, j_r=0}^{w_1-1} [j_1 + \cdots + j_r]_{q^{w_2}}^{n-i} q^{w_2(i+1) \sum_{l=1}^r j_l} \\
 & = \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i, q^{w_1}}^{(r)}(w_2 x) T_{n,i}^{(r)}(w_1 | q^{w_2}),
 \end{aligned}$$

where

$$(28) \quad T_{n,i}^{(r)}(w | q) = \sum_{j_1, \dots, j_r=0}^{w-1} [j_1 + \cdots + j_r]_q^{n-i} q^{(i+1) \sum_{l=1}^r j_l}.$$

By the same method as (28), we get

$$\begin{aligned}
 (29) \quad & [w_2]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r j_l} \\
 & \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y_1) \cdots d\mu_{q^{w_2}}(y_r) \\
 & = \sum_{i=0}^n \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^{i-r} \beta_{i, q^{w_2}}^{(r)}(w_1 x) T_{n,i}^{(r)}(w_2 | q^{w_1}).
 \end{aligned}$$

Therefore, by Corollary 2, (27) and (29), we obtain the following theorem.

Theorem 4. For $n \geq 0$ and $r, w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{i=0}^n \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^{i-r} \beta_{i,q^{w_2}}^{(r)}(w_1 x) T_{n,i}^{(r)}(w_2 | q^{w_1}) \\ &= \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i,q^{w_1}}^{(r)}(w_2 x) T_{n,i}^{(r)}(w_1 | q^{w_2}), \end{aligned}$$

where

$$T_{n,i}^{(r)}(w | q) = \sum_{j_1, \dots, j_r=0}^{w-1} [j_1 + \dots + j_r]_q^{n-i} q^{(i+1) \sum_{l=1}^r j_l}.$$

For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} [x + y_1 + \dots + y_r]_q^n q^{\sum_{l=1}^r (h-l)y_l} d\mu_q(y_1) \dots d\mu_q(y_r) \\ &= \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{q^{xj}}{(1-q)^n} \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q^r} \sum_{y_1, \dots, y_r=0}^{p^N-1} q^{j \sum_{l=1}^r y_l} q^{\sum_{l=1}^r (h-l+1)y_l} \\ &= \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{q^{xj}}{(1-q)^n} \frac{(j+h)(j+h-1) \dots (j+h-r+1)}{[j+h]_q [j+h-1]_q \dots [j+h-r+1]_q} \\ &= \frac{1}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j q^{xj} \frac{\binom{j+h}{r}_q}{\binom{j+h}{r}_q} \frac{r!}{[r]_q!}, \end{aligned}$$

where $\binom{x}{r}_q = \frac{[x]_q [x-1]_q \dots [x-r+1]_q}{[r]_q!} = \frac{[x]_q [x-1]_q \dots [x-r+1]_q}{[r]_q [r-1]_q \dots [2]_q [1]_q}.$

From (18), we can also define q -extensions of higher-order Bernoulli polynomials as follows :

$$(30) \quad \beta_{n,q}^{(h,r)}(x) = \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} [x + y_1 + \dots + y_r]_q^n q^{\sum_{l=1}^r (h-l)y_l} d\mu_q(y_1) \dots d\mu_q(y_r),$$

where $n \geq 0$ and $h \in \mathbb{Z}, r \in \mathbb{N}$.

Let $w_1, w_2 \in \mathbb{N}$. Then we see that

$$\begin{aligned} (31) \quad & \frac{1}{[w_1]_q^r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r (h-l+1)j_l} \\ & \times \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^r (h-l)y_l} e^{[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 y_l)]_q t} d\mu_{q^{w_1}}(y_1) \dots d\mu_{q^{w_1}}(y_r) \\ &= \frac{1}{[w_2]_q^r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r (h-l+1)j_l} \\ & \times \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{w_2 \sum_{l=1}^r (h-l)y_l} e^{[w_1 w_2 x + \sum_{l=1}^r (w_1 j_l + w_2 y_l)]_q t} d\mu_{q^{w_2}}(y_1) \dots d\mu_{q^{w_2}}(y_r). \end{aligned}$$

From (31), we have

$$\begin{aligned}
 (32) \quad & [w_1]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r (h-l+1) j_l} \\
 & \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^r (h-l) y_l} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
 = & [w_2]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r (h-l+1) j_l} \\
 & \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_2 \sum_{l=1}^r (h-l) y_l} \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y_1) \cdots d\mu_{q^{w_2}}(y_r),
 \end{aligned}$$

where $n \geq 0$ and $r \in \mathbb{N}$, $h \in \mathbb{Z}$.

Therefore, by (30) and (32), we obtain the following theorem.

Theorem 5. For $n \geq 0$, $h \in \mathbb{Z}$ and $w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}
 & [w_1]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r (h-l+1) j_l} \beta_{n, q^{w_1}}^{(h, r)} \left(w_2 x + \frac{w_2}{w_1} (j_1 + \cdots + j_r) \right) \\
 = & [w_2]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r (h-l+1) j_l} \beta_{n, q^{w_2}}^{(h, r)} \left(w_1 x + \frac{w_1}{w_2} (j_1 + \cdots + j_r) \right).
 \end{aligned}$$

From (30), we can derive the following equation :

$$\begin{aligned}
 (33) \quad & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^r (h-l) y_l} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
 = & \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i) \sum_{l=1}^r j_l} \\
 & \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^r (h-r) y_l} \left[w_2 x + \sum_{l=1}^r y_l \right]_{q^{w_1}}^{n-i} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
 = & \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i) \sum_{l=1}^r j_l} \beta_{n-i, q^{w_1}}^{(h, r)}(w_2 x).
 \end{aligned}$$

By (33), we get

$$\begin{aligned}
(34) \quad & [w_1]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r (h-l+1)j_l} \\
& \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^r (h-l)y_l} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
& = \sum_{j_1, \dots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r (h-l+1)j_l} \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} [j_1 + \cdots + j_r]_{q^{w_2}}^i \\
& \quad \times q^{w_2(n-i) \sum_{l=1}^r j_l} \beta_{n-i, q^{w_1}}^{(h,r)}(w_2 x) \\
& = \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} \beta_{n-i, q^{w_1}}^{(h,r)}(w_2 x) \sum_{j_1, \dots, j_r=0}^{w_1-1} [j_1 + \cdots + j_r]_{q^{w_2}}^i \\
& \quad \times q^{w_2 \sum_{l=1}^r (n+h-l-i+1)j_l} \\
& = \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i, q^{w_1}}^{(h,r)}(w_2 x) T_{n,i}^{(h,r)}(w_1 | q^{w_2}),
\end{aligned}$$

where

$$(35) \quad T_{n,i}^{(h,r)}(w|q) = \sum_{j_1, \dots, j_r=0}^{w-1} [j_1 + \cdots + j_r]_q^{n-i} q^{\sum_{l=1}^r (i+h-l+1)j_l}.$$

By the same method as (34), we see that

$$\begin{aligned}
(36) \quad & [w_2]_q^{n-r} \sum_{j_1, \dots, j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r (h-l+1)j_l} \\
& \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_2 \sum_{l=1}^r (h-l)y_l} \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_{q^{w_2}}(y_1) \cdots d\mu_{q^{w_2}}(y_r) \\
& = \sum_{i=0}^n \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^{i-r} \beta_{i, q^{w_2}}^{(h,r)}(w_1 x) T_{n,i}^{(h,r)}(w_2 | q^{w_1}).
\end{aligned}$$

Therefore, by (34) and (36), we obtain the following theorem.

Theorem 6. For $n \geq 0$, $h \in \mathbb{Z}$ and $r, w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}
& \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i, q^{w_1}}^{(h,r)}(w_2 x) T_{n,i}^{(h,r)}(w_1 | q^{w_2}) \\
& = \sum_{i=0}^n \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^{i-r} \beta_{i, q^{w_2}}^{(h,r)}(w_1 x) T_{n,i}^{(h,r)}(w_2 | q^{w_1}),
\end{aligned}$$

where

$$T_{n,i}^{(h,r)}(w|q) = \sum_{j_1, \dots, j_r=0}^{w-1} [j_1 + \cdots + j_r]_q^{n-i} q^{\sum_{l=1}^r (h+i-l+1)j_l}.$$

Remark. A p -adic approach to identities of symmetry for Carlitz's q -Bernoulli polynomials has been studied in [?].

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